Inversion of the Van der Monde Matrix

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***Abstract*— In this letter we deduce an analytical formula for inversion of a complex Van der Monde matrix. It has applications in signal reconstruction, spectral estimation, system identification, as well as in other important signal processing problems. Since till now the inversion of such a matrix has been performed by approximations, we further deduce an exact inversion formula.**

I. Introduction

THERE have been several approaches to define a *nonuniform discrete Fourier transform (NDFT)*, the most recent ones belonging to Neagoe [3], [4], [5] and Mitra *et al.* [2]; the definition of the NDFT requires the inversion of a complex Van der Monde matrix. Signal reconstruction [3], system identification and other important signal processing problems necessitate also such inversion. Till now, the inversion of the Van der Monde matrix has been performed by approximations. We further deduce an exact formula for inversion.

II. Preliminary Notations

Consider a Van der Monde matrix of order n over $\boldsymbol{C}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}, \qquad a_k \in C. \tag{1}$$

Denote by

$$\det A = V_n(a_1, \cdots, a_n). \tag{2}$$

We know that

$$V_n(a_1, \cdots, a_n) = \prod_{n \geq i > j \geq 1} (a_i - a_j). \tag{3}$$

For $1 \leq k \leq n-1$, consider the determinant

$$V_n^k(a_1, \cdots, a_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_n^{k-1} \\ a_1^{k+1} & a_2^{k+1} & \cdots & a_n^{k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix}. \tag{4}$$

For $k = 0$, we define

$$V_n^0(a_1, \cdots, a_n) = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} \tag{5}$$

and for $k = n$, we also define

$$V_n^n(a_1, \cdots, a_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}. \tag{6}$$

Obviously

$$V_n^n(a_1, \cdots, a_n) = V_n(a_1, \cdots, a_n). \tag{7}$$

III. Deduction of the Inversion Formula

Consider the Van der Monde determinant of order $(n+1)$, i.e.

$$V_{n+1}(a_1, \cdots, a_n, z) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & z \\ a_1^2 & a_2^2 & \cdots & a_n^2 & z^2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1^n & a_2^n & \cdots & a_n^n & z^n \end{vmatrix} \tag{8}$$

where z is a complex variable.

We can easily prove that

$$V_{n+1}(a_1, a_2, \cdots, a_n, z) = V_n(a_1, a_2, \cdots, a_n) \cdot \prod_{i=1}^{n} (z - a_i). \tag{9}$$

On the other side, if we develop $V_{n+1}(a_1, a_2, \cdots, a_n, z)$ after its last column, we yield

$$V_{n+1}(a_1, a_2, \cdots, a_n, z) = V_n^n(a_1, a_2, \cdots, a_n.)z^n - V_n^{n-1}(a_1, a_2, \cdots, a_n)z^{n-1} + \cdots + (-1)^n V_n^0(a_1, a_2, \cdots, a_n). \tag{10}$$

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$$A^{-1} = \left((-1)^{i+j} \frac{\sigma_{n-i}(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n)}{\prod_{k=1}^{j-1} (a_j - a_k) \prod_{k=j+1}^{n} (a_k - a_j)} \right)^T_{i=1,\cdots,n;\, j=1,\cdots,n} \tag{17}$$

$$A^{-1} = \begin{pmatrix} \frac{\sigma_{n-1}(a_2, a_3, \cdots, a_n)}{\prod_{k=2}^{n} (a_k - a_1)} & -\frac{\sigma_{n-2}(a_2, a_3, \cdots, a_n)}{\prod_{k=2}^{n} (a_k - a_1)} & \cdots & (-1)^{n+1} \frac{1}{\prod_{k=2}^{n} (a_k - a_1)} \\ -\frac{\sigma_{n-1}(a_1, a_3, \cdots, a_n)}{(a_2 - a_1) \prod_{k=3}^{n} (a_k - a_2)} & \frac{\sigma_{n-2}(a_1, a_3, \cdots, a_n)}{(a_2 - a_1) \prod_{k=3}^{n} (a_k - a_2)} & \cdots & (-1)^{n+2} \frac{1}{(a_2 - a_1) \prod_{k=3}^{n} (a_k - a_2)} \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{n+1} \frac{\sigma_{n-1}(a_1, a_2, \cdots, a_{n-1})}{\prod_{k=1}^{n-1} (a_n - a_k)} & (-1)^{n+2} \frac{\sigma_{n-2}(a_1, a_2, \cdots, a_{n-1})}{\prod_{k=1}^{n-1} (a_n - a_k)} & \cdots & \frac{1}{\prod_{k=1}^{n-1} (a_n - a_k)} \end{pmatrix} \tag{18}$$

Consider the symmetrical polynomials of degree k, $0 \le k \le n$, having as variables $a_1, a_2, \cdots, a_n$, namely

$$\begin{cases} \sigma_0(a_1, a_2, \cdots, a_n) = 1 \\ \sigma_1(a_1, a_2, \cdots, a_n) = a_1 + a_2 + \cdots + a_n \\ \sigma_2(a_1, a_2, \cdots, a_n) = a_1a_2 + a_1a_3 + \cdots + a_1a_n \\ \qquad + a_2a_3 + \cdots + a_2a_n + \cdots + a_{n-1}a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_n(a_1, a_2, \cdots, a_n) = a_1a_2\cdots a_n. \end{cases} \tag{11}$$

Taking into account that

$$\prod_{i=1}^{n} (z - a_i) = \sigma_0(a_1, a_2, \cdots, a_n)z^n - \sigma_1(a_1, a_2, \cdots, a_n) \cdot z^{n-1} + \cdots + (-1)^n \sigma_n(a_1, a_2, \cdots, a_n) \tag{12}$$

relations (9)–(12) lead to

$$V_n^k(a_1, a_2, \cdots, a_n) = V_n(a_1, a_2, \cdots, a_n)\sigma_{n-k}(a_1, a_2, \cdots, a_n), \quad 0 \le k \le n. \tag{13}$$

Denote by A_{ij} the algebraic complement of the element a_j^{i-1} placed on the ith row and jth column of the Van der Monde matrix $\boldsymbol{A}$, $(1 \le j \le n,\ 1 \le i \le n)$. We have

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_j & \cdots & a_n \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ a_1^{i-1} & a_2^{i-1} & \cdots & a_j^{i-1} & \cdots & a_n^{i-1} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_j^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$$

$$= (-1)^{i+j} V_{n-1}^{i-1}(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n). \tag{14}$$

Taking into account relation (13), we obtain

$$A_{ij} = (-1)^{i+j} V_{n-1}(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n) \cdot \sigma_{n-i}(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n). \tag{15}$$

It leads to

$$\frac{A_{ij}}{V_n(a_1, a_2, \cdots, a_n)} = (-1)^{i+j} \frac{\sigma_{n-i}(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n)}{\prod_{k=1}^{j-1} (a_j - a_k) \prod_{k=j+1}^{n} (a_k - a_j)}. \tag{16}$$

Since then, the inverse matrix is expressed in (17), shown at the top of the page, where T denotes the transposition.

It can be equivalently expressed in (18), which appears at the top of the page.

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